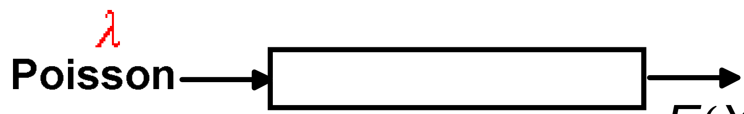
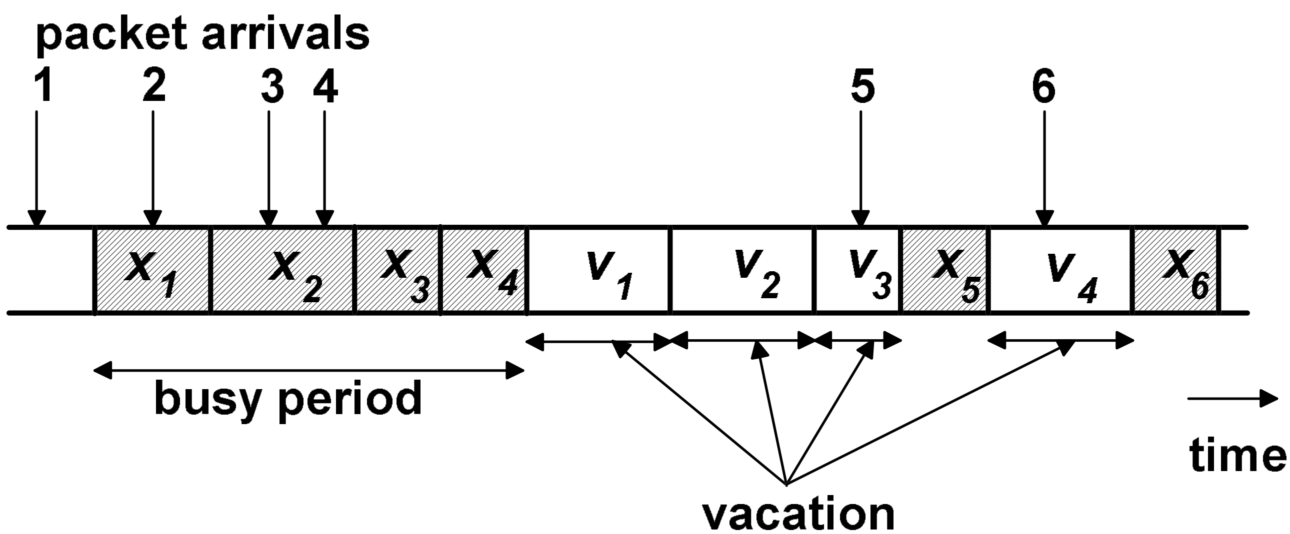
M/G/1 with vacations



$E(X) = \frac{1}{\mu},\ E(X^2)$

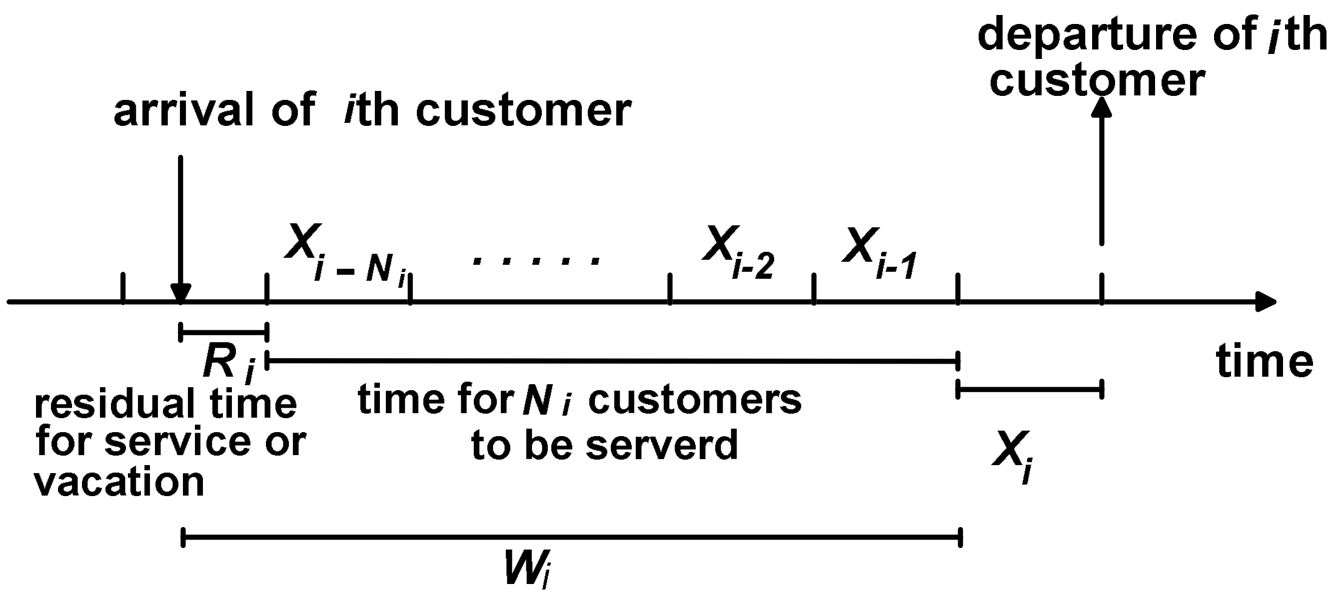
Arrivals and Service times are independent

Whenever a busy period finishes, the system goes "on vacation" for some random interval of duration V.



Vacation times are independently and identically distributed with known $E(V)$ and $E(V^2)$.

They are also independent of arrival times and service times.

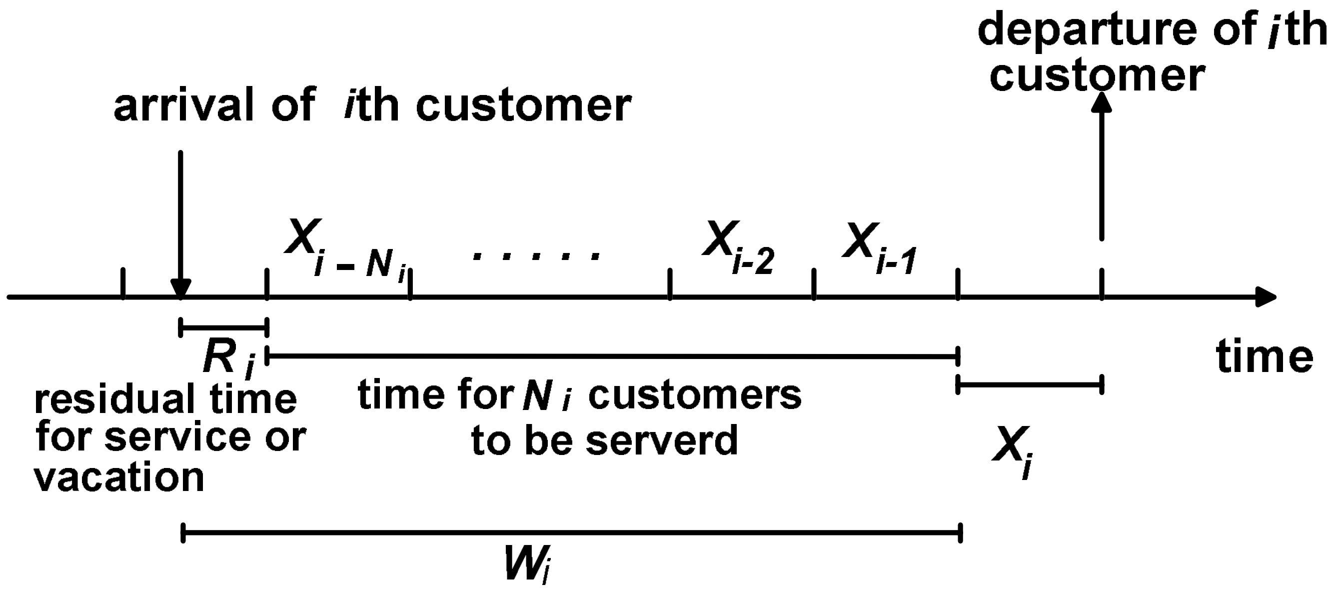


W_i = **Waiting time in the queue of the** *i*th **customer**

R_i = **Residual service or vacation time seen by the** *i*th **customer. By this we mean that if customer** *j* **is already being served when** *i* **arrives,** R_i **is the remaining time until customer** *j* **'s service time is complete. If no customer is in service (i.e., the system is empty when** *i* **arrives), then** R_i **is zero**

X_i = **service time of the** *i*th **customer**

N_i = **Number of customers found waiting in queue by** *i*th **customer upon arrival** *i* **(assume FCFS, even though not necessary)**



Formula $W = \frac{R}{1-\rho}$ **still holds,**

but $R = \frac{\lambda E(X^2)}{2} + \frac{(1-\rho)\cdot E(V^2)}{2E(V)}$

$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$\Rightarrow E(W_i) = E(R_i) + E(\sum_{j=i-N_i}^{i-1} X_j)$$

$$\Rightarrow E(W_i) = E(R_i) + E(N_i)E(X)$$

$i \to \infty$

$$W = R + N_Q \cdot \frac{1}{\mu}$$

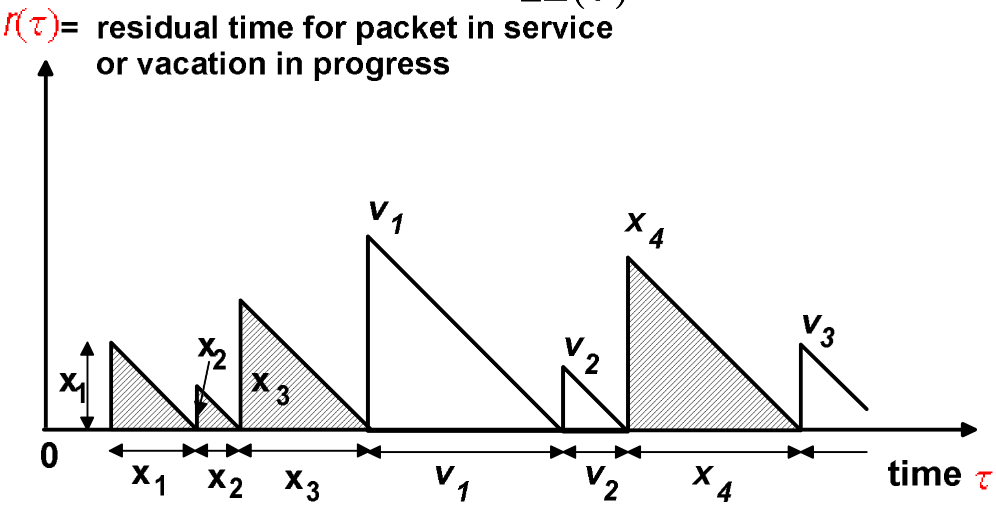
$N_Q = \lambda W$

$$\Rightarrow W = R + \lambda W \cdot \frac{1}{\mu} = R + \frac{\lambda}{\mu} W$$

$\frac{\lambda}{\mu} = \rho$

$$\Rightarrow W = \frac{R}{1-\rho}$$

Proof of $R = \frac{\lambda E(X^2)}{2} + \frac{(1-\rho)\cdot E(V^2)}{2E(V)}$



Let $M(t)$ **= # of service completed by time** t

$L(t)$ **= # of vacation completed by time** t

Time average of $r(\tau)$ **up to time** t

$$= \frac{1}{t}\int_0^t r(\tau)d\tau = \frac{1}{t}\left(\sum_{i=1}^{M(t)} \frac{1}{2}X_i^2 + \sum_{i=1}^{L(t)} \frac{1}{2}V_i^2\right)$$

$$= \frac{1}{2}\cdot\underbrace{\frac{M(t)}{t}}_{=\lambda}\cdot\underbrace{\frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)}}_{=E(X^2)} + \frac{1}{2}\cdot\underbrace{\frac{L(t)}{t}}_{=\frac{1-\rho}{E(V)}}\cdot\underbrace{\frac{\sum_{i=1}^{L(t)} V_i^2}{L(t)}}_{=E(V^2)}$$

$t \to \infty$

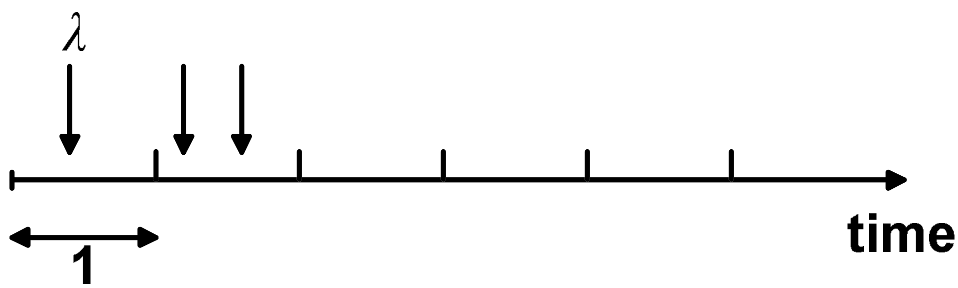
$$(1-\rho)t = L(t)\cdot E(V) \quad \Rightarrow \frac{L(t)}{t} = \frac{1-\rho}{E(V)}$$

$$\Rightarrow R = \frac{\lambda E(X^2)}{2} + \frac{(1-\rho)\cdot E(V^2)}{2E(V)}$$

$$W = \frac{R}{1-\rho} = \underbrace{\frac{\lambda E(X^2)}{2(1-\rho)}}_{} + \underbrace{\frac{E(V^2)}{2E(V)}}_{}$$

***M/G/1* delay** **additional delay due to vacations**

Example: slotted *M/D/1* system



- **All packets has constant service time = 1 unit of time**
- **Service of packet can begin only at the beginning of a slot**

$$E(X) = E(X^2) = 1$$

$$E(V) = E(V^2) = 1$$

$$W = \frac{\lambda E(X^2)}{2(1-\rho)} + \frac{E(V^2)}{2E(V)} = \underbrace{\frac{\lambda}{2(1-\lambda)}}_{} + \frac{1}{2}$$

M/D/1 **without slots**

Example 3.16

m **traffic streams of equal-length packets.**

Poisson arrival with each rate $\frac{\lambda}{m}$

FDM: service time = *m*

M/D/1 **with** $\frac{\lambda}{m}$ $,\mu = \frac{1}{m}$

$$W_{FDM} = \frac{\lambda \cdot \overline{X^2}}{2(1-\rho)} = \frac{\lambda \cdot \overline{X}^2}{2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)} = \frac{\lambda m}{2(1-\lambda)}$$

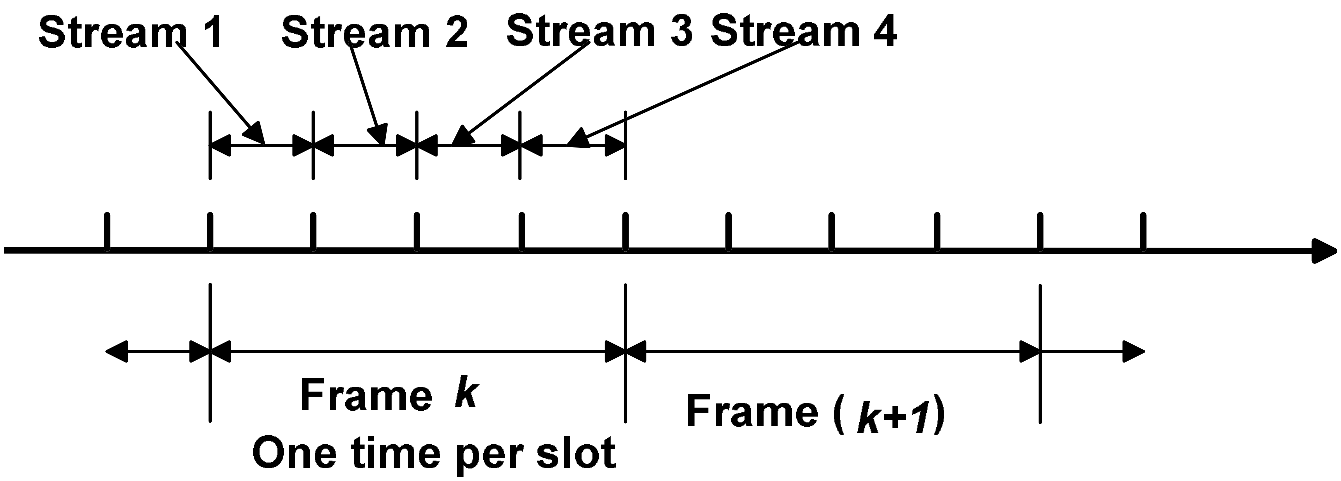
Slotted FDM: packet transmissions start at times *m*, *2m*, *3m*,......

M/D/1 **with vacations**

$$\overline{V} = m, \overline{V^2} = m^2$$

$$W_{SFDM} = W_{FDM} + \frac{\overline{V^2}}{2\overline{V}} = W_{FDM} + \frac{m}{2} = \frac{m}{2(1-\lambda)}$$

TDM with m = 4 traffic streams



$$W_{TDM} = W_{SFDM}$$

$$T_{FDM} = m + W_{FDM}$$

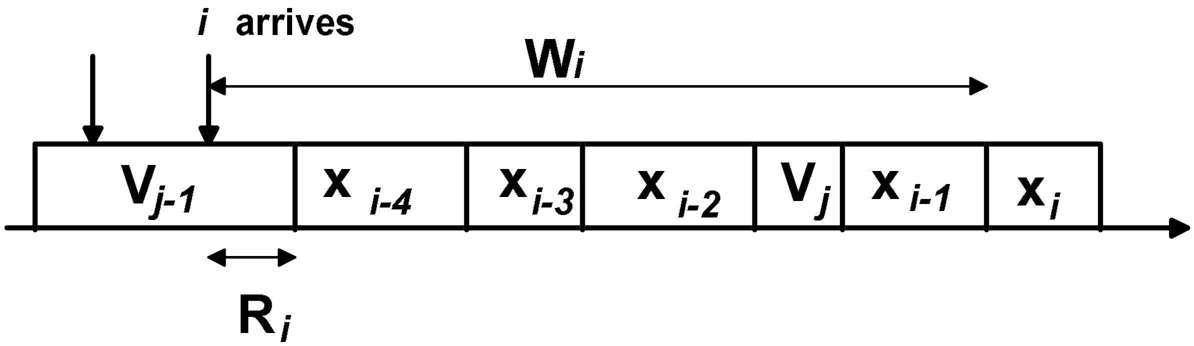
$$T_{SFDM} = \frac{3m}{2} + W_{FDM}$$

$$T_{TDM} = 1 + \frac{m}{2} + W_{FDM}$$

Reservation Systems

Reservation system with a single class of users

- **Data intervals and reservation intervals alternate.**
- **All packets arriving during one data interval and preceding reservation interval wait until the next reservation interval to make a reservation ("gated" version)**
- **reservation intervals are iid and independent of arrivals and service time**



$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j + V_j$$

$$\Rightarrow W = R + \underbrace{N_Q}_{\lambda \cdot W} \cdot E(X) + E(V)$$

$$\rho = \lambda \cdot E(X)$$

$$\Rightarrow W = \frac{R + E(V)}{1-\rho}$$

R **is the same as in *M/G/1* with vacations**

The residual service time is the same as in the vacation case, so

$$W = \underbrace{\frac{\lambda(E(X^2)}{2(1-\rho)} + \frac{E(V^2)}{2E(V)}}_{\textbf{\textit{M/G/1} vacations}} + \frac{E(V)}{1-\rho}$$

If all reservation intervals are of constant duration *V***,**

$$W = \frac{\lambda \cdot E(X^2)}{2(1-\rho)} + V \cdot \left(\frac{1}{2} + \frac{1}{1-\rho}\right)$$