## *M/M/1* system:

- Poisson arrivals
- Exponential service times
- Service times mutually independent and independent of arrival times

## *M/M/m* system:

- Like *M/M/1*, but now we have *m* servers

## M/M/$\infty$ system:

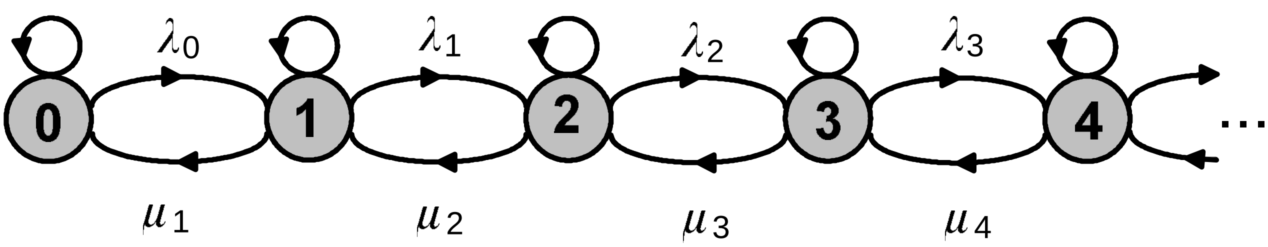
- servers

## *M/M/m/m* system:

- Like *M/M/m*, but customers arriving when all servers are busy are lost

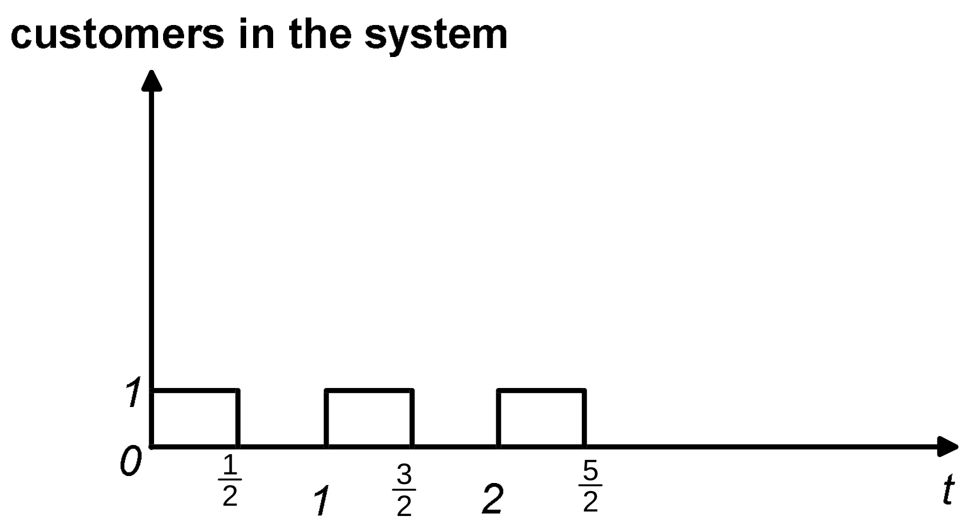
# Birth-death Markov chains

**A birth-death Markov chain is a Markov chain with integers as states, and with transitions only between neighboring states (e.g. *M/M/1*, *M/M/m*, *M/M/m/m*, *M/M/*$\infty$ ).**



**Example: *D/D/1* system (deterministic)**

**Assume interarrival times** = *1*
**Assume service times** = $\frac{1}{2}$




**Arrivals occur at times** $0, 1, 2, \cdots$

**Departure occur at times** $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$

**A customer always find the system empty when it arrives.**

$$N = \frac{1}{2}$$

## Occupancy distribution upon arrival/departure

$p_n(t) = P_r(N(t) = n) \rightarrow$ **probability that there are** $n$ **customers in the system at time** $t$ **.**

$p_n = \lim_{t\to\infty} p_n(t) \rightarrow$ **steady-state probability that there are** $n$ **customers in the system**

**Let**

$a_n(t) = p_r\{N(t) = n \mid$ **an arrival ocurred just after time** $t$ **}**

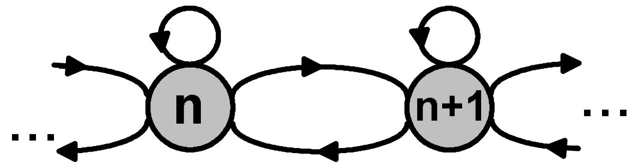
$a_n = \lim_{t\to\infty} a_n(t) \rightarrow$ **steady-state probability that an arriving customer finds** $n$ **customers in the system**

$d_n(t) = p_r\{N(t) = n \mid$ **an departure ocurred just before time** $t$ **}**

$d_n = \lim_{t\to\infty} d_n(t) \rightarrow$ **steady-state probability that an departing customer leaves** $n$ **customers behind**

**Fact 1**: under broad assumptions, $a_n = d_n$

**Proof:**




**Intuitive reason**: on any sample path, number of arrivals seeing state $n$ is number of $n \to n+1$ transitions; this equals number of $n+1 \to n$ transitions $(\pm 1)$, which is a number of departure leaving system in state $n$.

frequency of $n \to n+1$ transitions
= frequency of $n+1 \to n$ transitions

**Fact 2**: for Poisson arrivals, and service times that are independent of interarrival times.

$$\boxed{a_n = p_n (= d_n)}$$

**Proof: let** $A(t, t+\delta)$ **be the event that an arrival occurs between time** $t$ **and** $t+\delta$

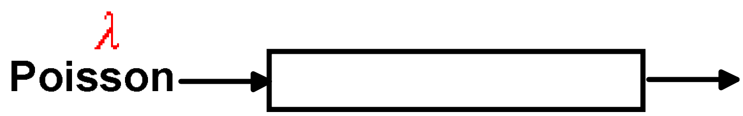
$$a_n(t) = \lim_{\delta \to 0} P_r\{N(t) = n \mid A(t, t+\delta) \text{ occured}\}$$ **Bayes rule**

$$\Rightarrow a_n(t) = \lim_{\delta \to 0} \frac{P_r\{A(t,t+\delta) \mid N(t)=n\} \cdot Pr\{N(t)=n\}}{p_r\{A(t,t+\delta)\}}$$

$$= P_r\{N(t) = n\} = p_n(t) \Rightarrow a_n = p_n$$

**i.e., in steady-state, "arrivals see steady-state probabilities".**

# *M/G/1* queue




- **Poisson arrivals with rate $\lambda$**
- **Service time follow an arbitrary distribution with given** $E(X) = \frac{1}{\mu}$ **and** $E(X^2)$
- **Service times are iid, and independent of arrivals**

# P-K formula

$$W = \frac{\lambda \cdot E(X^2)}{2(1-\rho)}$$

**where** $\rho = \lambda E(X) = \frac{\lambda}{\mu}$ **(server utilization factor)**

**From Little's theorem:** $N_Q = \lambda W,\ T = E(X) + W,\ N = \lambda T$

**The proof of the P-K formula will use graphical arguments.**

**Example 1** ***(M/M/1)***

$$E(X) = \frac{1}{\mu}\ ,\ E(X^2) = \frac{2}{\mu^2}$$

**P-K** $\Rightarrow W = \frac{\lambda(\frac{2}{\mu^2})}{2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}$

**<u>Example 2</u> (*M/D/1*)**

**Service times are deterministic and equal to** $E(X) = \frac{1}{\mu}$**.**

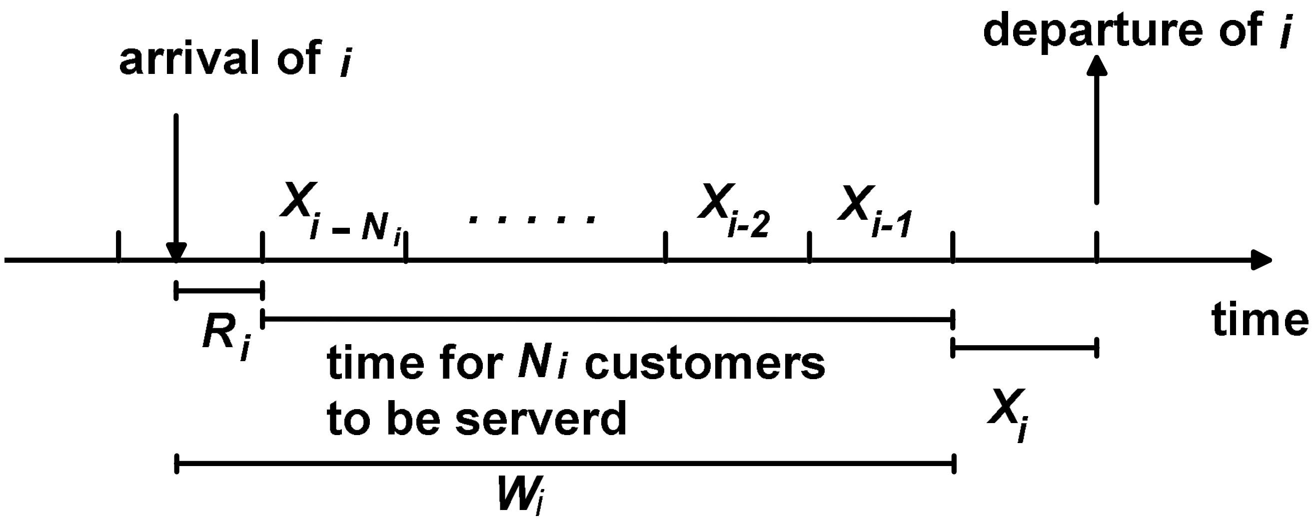
$$E(X^2) = \text{var}\,(X) + (E(X))^2 \qquad \text{var}\,(X) = 0$$

$$E(X^2) = (E(X))^2 = \frac{1}{\mu^2}$$

$$\overset{P-K}{\Rightarrow}\; W = \frac{\lambda(\frac{1}{\mu^2})}{2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}$$

**Note: *M/D/1* has the smallest** $N$**,** $T$**,** $N_Q$**,** $W$ **over all *M/G/1* systems with the same** $E(X) = \frac{1}{\mu}$

# Proof of the P-K formula

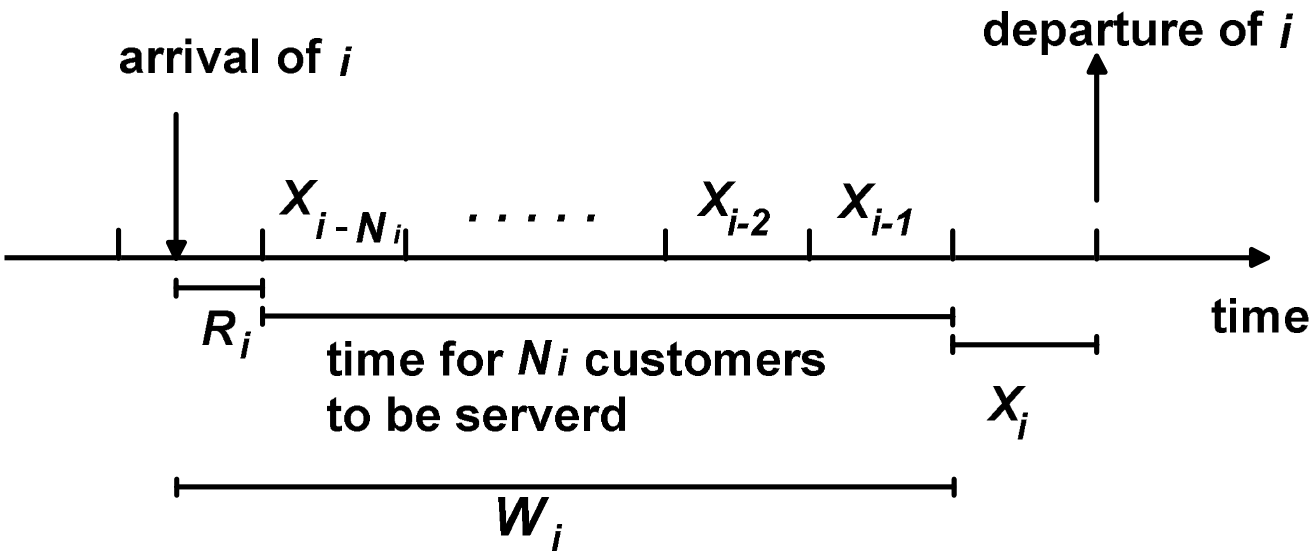



$W_i$ = **Waiting time in the queue of the** $i$th **customer**

$R_i$ = **Residual service time seen by the** $i$th **customer. By this we mean that if customer** $j$ **is already being served when** $i$ **arrives,** $R_i$ **is the remaining time until customer** $j$ **'s service time is complete. if no customer is in service (i.e., the system is empty when** $i$ **arrives), then** $R_i$ **is zero**

$X_i$ = **service time of the** $i$th **customer**

$N_i$ = **Number of customers found waiting in queue by** $i$th **customer upon arrival** $i$ **(assume FCFS, even though not necessary**




$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$\Rightarrow E(W_i) = E(R_i) + E\left(\sum_{j=i-N_i}^{i-1} X_j\right)$$

$$\Rightarrow E(W_i) = E(R_i) + E(N_i)E(X_j)$$

$i \to \infty$

$$W = R + N_Q \cdot \frac{1}{\mu}$$

$N_Q = \lambda W$

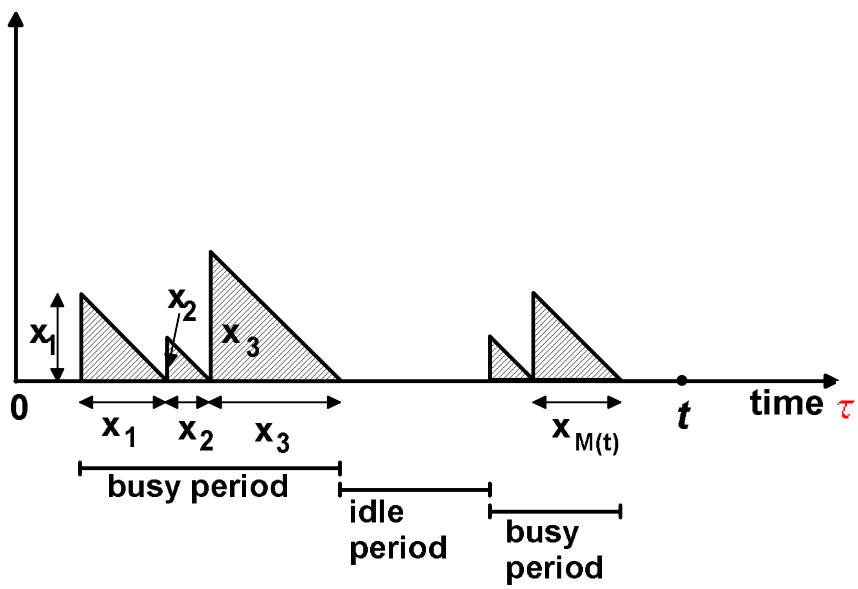
$$\Rightarrow W = R + \lambda W \cdot \frac{1}{\mu} = R + \frac{\lambda}{\mu} W$$

$\frac{\lambda}{\mu} = \rho$

$$\boxed{\Rightarrow W = \frac{R}{1-\rho}}$$

**We will show that** $R = \dfrac{\lambda E(X^2)}{2}$

$r(\tau)$ = residual time of customer in service




*M(t)* = **number of service completions up to time** *t*.

**Time average of** $r(\tau)$ **up to time** $t$

$$= \frac{1}{t}\int_0^t r(\tau)d\tau = \frac{1}{t}\sum_{i=1}^{m(t)} \frac{1}{2}X_i^2 = \frac{1}{2}\cdot\underbrace{\frac{M(t)}{t}}_{=\lambda}\cdot\underbrace{\frac{\sum_{i=1}^{m(t)} X_i^2}{M(t)}}_{=E(X^2)}$$

$$\overset{t\to\infty}{\Rightarrow} \quad R = \frac{\lambda E(X^2)}{2}$$

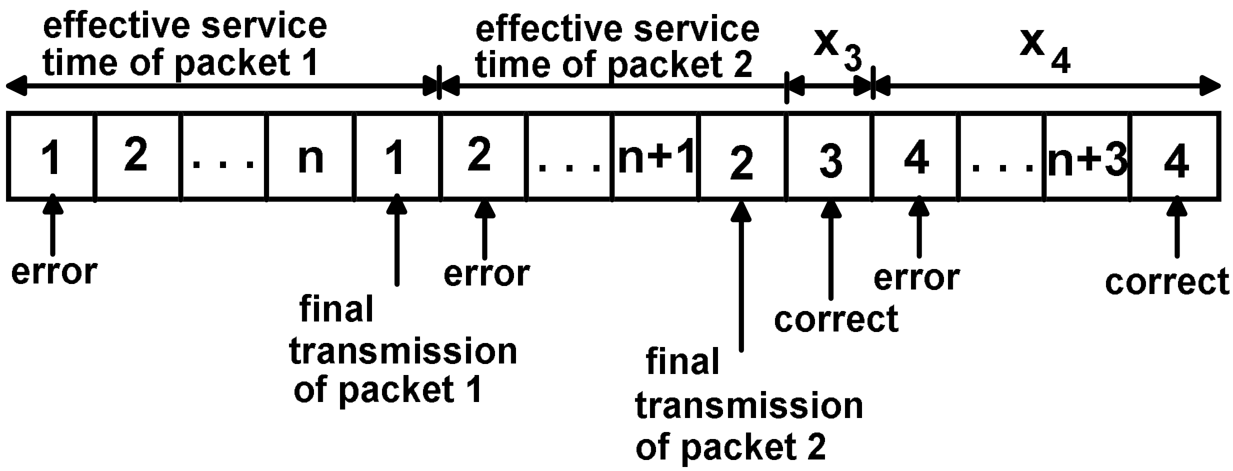
**Note: for stability** $\rho = \lambda E(X) = \frac{\lambda}{\mu} < 1$ and $E(X^2) < \alpha$

Example: delay analysis of a go back n ARQ

**Assume retransmissions happen only due to errors in forward channel.**

***p*** **= probability that a frames is received with errors.**

***X*** **= effective service time of a packet (time between its first and last transmissions)**




**packets arrive at transmitter according to Poisson process of rate $\lambda$**


$$P(X = 1 + kn) = (1-p)\cdot p^k$$

$$E(X) = \sum_{k=0}^{\infty} (1+kn)(1+p)\cdot p^K = \cdots = 1 + \frac{np}{1-p}$$

$$E(X^2) = \sum_{k=0}^{\infty} (1+kn)^2(1+p)\cdot p^K = \cdots = 1 + \frac{2np}{1-p} + \frac{n^2(p+p^{2)}}{(1-p)^2}$$

**P-K**: $W = \frac{\lambda E(X^2)}{2[1-\lambda E(X)]}$ , $T = E(X) + W$