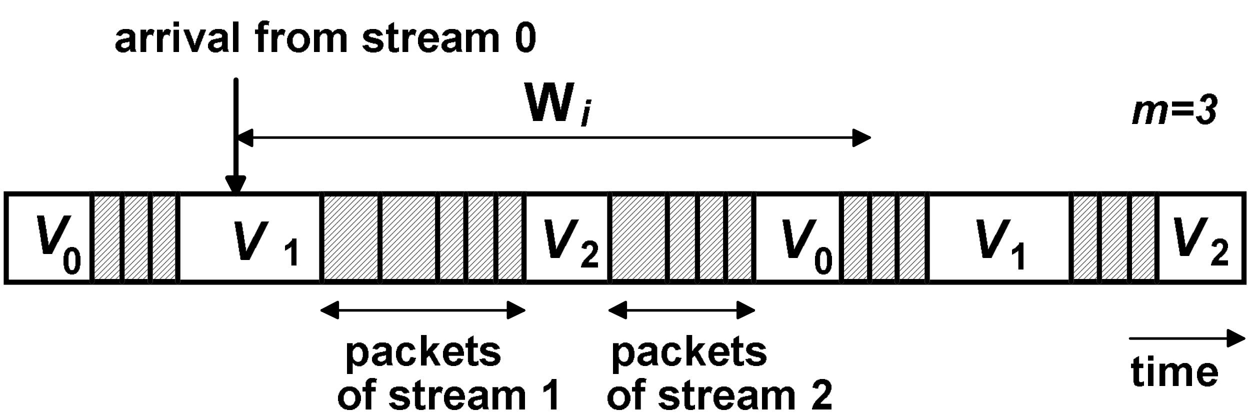
# Multiuser Reservation System

**Consider system with** $m$ **incoming streams, each Poisson with rate** $\frac{\lambda}{m}$**, service time** $X_n$ **are iid with** $E(X) = \frac{1}{\mu}$ , $E(X^2)$ **known, and independent of arrivals.**

**Server serves all packets from stream** *0***, then all from stream** *1,....,* **then all from** *m-1***, then all from** *0***, etc..**

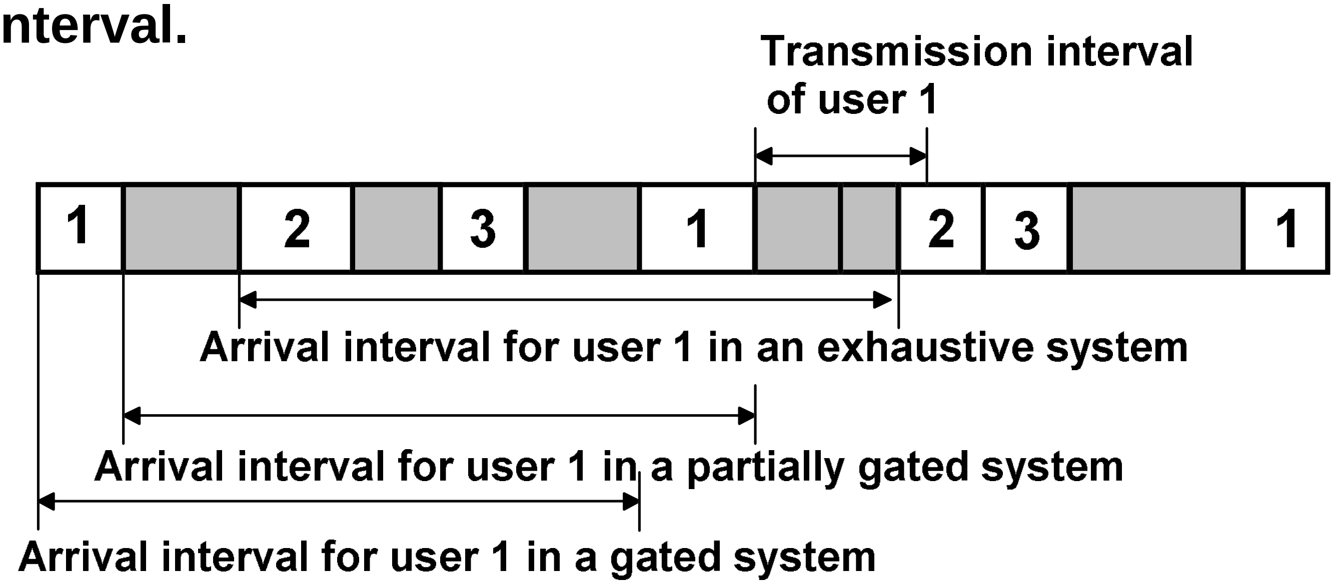
**There is a reservation interval of duration** $V_i$ **before transmission of packets from stream** *i***.**



**Gated service: all stream *i* arrivals during reservation or service interval for *i* are queued until the next stream *i* service interval.**

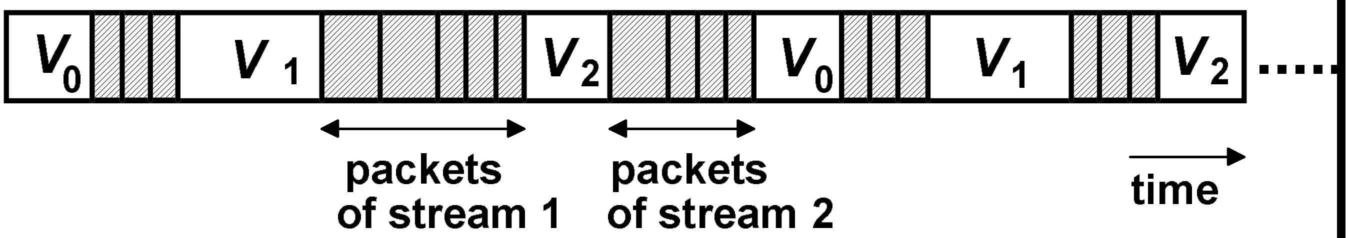
**Partially gated service: all stream *i* arrivals during reservation period for *i* get transmitted, but arrivals during service interval for *i* are queued for the next stream *i* interval.**

**Exhaustive service: all stream *i* arrivals while the server is serving *i* get transmitted in that service interval.**




**Packets arriving in the arrival shown are transmitted in the transmission interval shown**

**Assume exhaustive service:**




$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j + Y_i \;,$$

**where $Y_i$ is the duration of all the whole reservation intervals during which packet *i* must wait before being transmitted.**

$$R = \frac{\lambda \overline{X^2}}{2} + \frac{(1-\rho)\sum_{l=0}^{m-1}\overline{V_l^2}}{2\sum_{l=0}^{m-1}\overline{V_l}}$$

**From Little's theorem (even though service is not FCFS)**

$$E(N_i) = \lambda \cdot E(W_i)$$

**So** $W = \frac{R+Y}{1-\rho}$

$$Y = \frac{\rho \overline{V}(m-1)}{2} + \frac{m\overline{V}(1-\rho)}{2} - \frac{(1-\rho)\sum_{l=0}^{m-1}\overline{V_l}^2}{2m\overline{V}} = \frac{\overline{V}(m-\rho)}{2} - \frac{(1-\rho)\sum_{l=0}^{m-1}\overline{V_l}^2}{2m\overline{V}}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V}(m-\rho)}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}}$$ **where** $$\sigma_V^2 = \frac{\sum_{l=0}^{m-1}\left(\overline{V_l^2} - \overline{V_l}^2\right)}{m}$$

## To calculate Y for an exhaustive system

$\alpha_{lj} = E\{Y_i$ | **packet *i* arrives in user *l* 's reservation or data interval and belongs to user *(l+j)* mod *m* }**

$$\alpha_{lj} = \begin{cases} 0, & j=0 \\ \overline{V}_{(l+1) \bmod m} + \cdots + \overline{V}_{(l+j) \bmod m}, & j>0 \end{cases}$$

**Packet *i* belongs to any user with equal probability** $\frac{1}{m}$

$E\{Y_i$ | **packet *i* arrives in user *l* 's reservation or da interval** $\}$

$$= \frac{1}{m} \sum_{j=1}^{m-1} \alpha_{lj} = \sum_{j=1}^{m-1} \frac{m-j}{m} \overline{V}_{(l+j) \bmod m}$$

**A packet will arrive during user *l* 's data interval with probability** $\frac{\rho}{m}$**, and during user *l* 's reservation interval with probability**$\frac{(1-\rho)\overline{V}_l}{\sum_{k=0}^{m-1} \overline{V}_k}$**.**

$i \to \infty$

$$Y = \sum_{l=0}^{m-1} \{ [\frac{\rho}{m} + \frac{(1-\rho)\overline{V}_l}{\sum_{k=0}^{m-1} \overline{V}_k}] \sum_{j=1}^{m-1} \frac{m-j}{m} \overline{V}_{(l+j) \bmod m} \}$$

$$= \frac{\rho}{m} \sum_{j=1}^{m-1} \frac{m-j}{m} ( \sum_{l=0}^{m-1} \overline{V}_l ) + \frac{(1-\rho)\overline{V}_l}{\sum_{k=0}^{m-1} \overline{V}_k} \sum_{k=0}^{m-1} \sum_{j=1}^{m-1} \frac{m-j}{m} \overline{V}_l \overline{V}_{(l+j) \bmod m}$$

**The last sum**

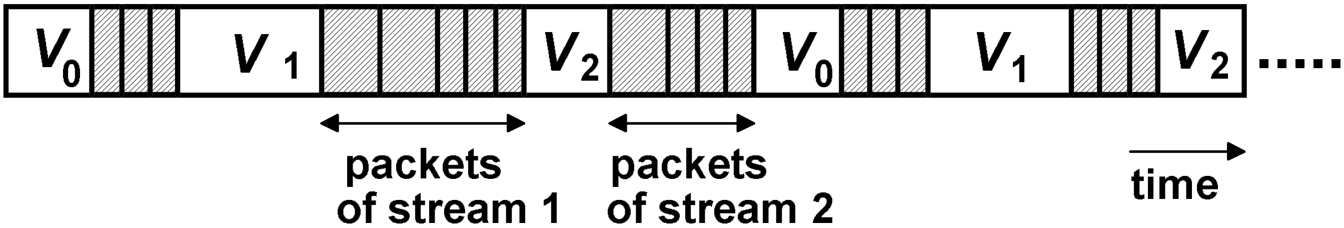
$$\sum_{k=0}^{m-1}\sum_{j=1}^{m-1} \frac{m-j}{m}\overline{V}_l\,\overline{V}_{(l+j)\bmod m} = \frac{1}{2}\Big[\Big(\sum_{k=0}^{m-1}\overline{V}_l\Big)^2 - \sum_{k=0}^{m-1}\overline{V}_l^{\,2}\Big]$$

**The sum of all possible products** $\overline{V}_l\,\overline{V}_{l'},\ l \neq l'$.

**Let** $\overline{V} = \frac{1}{m}\sum_{l=0}^{m-1}\overline{V}_l$

$$Y = \frac{\rho\overline{V}(m-1)}{2} + \frac{m\overline{V}(1-\rho)}{2} - \frac{(1-\rho)\sum_{l=0}^{m-1}\overline{V}_l^{\,2}}{2m\overline{V}} = \frac{\overline{V}(m-\rho)}{2} - \frac{(1-\rho)\sum_{l=0}^{m-1}\overline{V}_l^{\,2}}{2m\overline{V}}$$

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \frac{\overline{V}(m-\rho)}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}}$$ **where** $$\sigma_V^2 = \frac{\sum_{l=0}^{m-1}(\overline{V_l^2} - \overline{V}_l^{\,2})}{m}$$




**For partially gated systems**

**Prob. a user arrives during its own transmission interval =** $\frac{\rho}{m}$

**Y is increased by** $\frac{\rho}{m} \times m\overline{V} = \rho\overline{V}$

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \frac{(m+\rho)\overline{V}}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}}$$

**For fully gated systems**

**Prob. a user arrives during its own reservation or transmission interval =** $\frac{1}{m}$

**Y is increased by** $\frac{1}{m} \times m\overline{V} = \overline{V}$

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \frac{(m+2-\rho)V}{2(1-\rho)} + \frac{\sigma_V^2}{2\overline{V}}$$

# Nonpreemptive Priority Systems

- **Priority classes K = 1, 2, ....., n**
  **highest lowest**
- **Poisson arrivals with rate** $\lambda_k$**, general service times** $E(X_k) = \frac{1}{\mu}$ **and** $E(X_k^2)$ **given**
- **Arrival processes independent and independent of service time.**

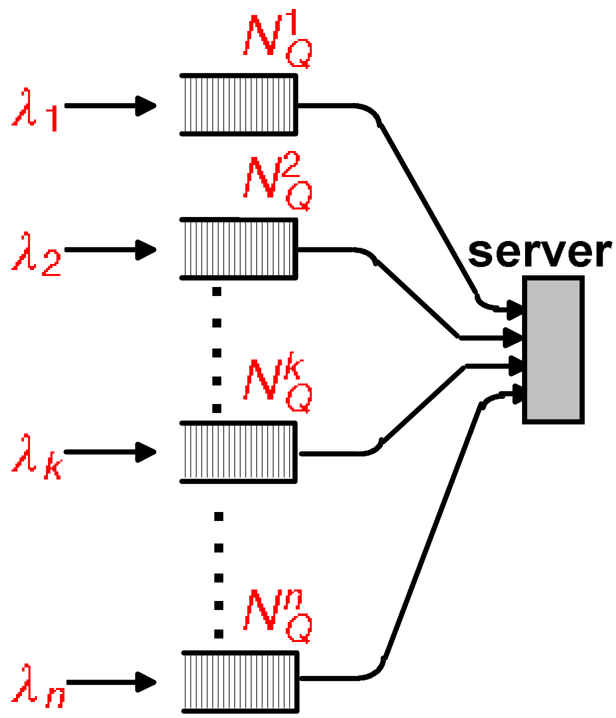
- **Nonpreemptive Priority:**
  - **Customer currently in service is allowed to complete service.**
  - **When server free, customer of highest priority class enters service.**

$N_Q^k =$ **Average number in queue for priority** $k$

$W_k =$ **Average queueing time for priority** $k$

$\rho_k = \frac{\lambda_k}{\mu_k} =$ **System utilization for priority** $k$

$R =$ **Mean residual service time**



**For highest priority class:**

$$\left.\begin{array}{l} W_1 = R + \frac{N_Q^1}{\mu_1} \\ N_Q^1 = \lambda_1 \cdot W_1 \end{array}\right\} \Rightarrow W_1 = R + \rho_1 W_1 \Rightarrow \boxed{W_1 = \frac{R}{1-\rho_1}}$$

**For class 2:**

$$\left.\begin{array}{l} W_2 = R + \frac{N_Q^1}{\mu_1} + \frac{N_Q^2}{\mu_2} + \frac{1}{\mu_1} \times \lambda_1 W_2 \\ N_Q^1 = \lambda_1 W_1, N_Q^2 = \lambda_2 W_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow W_2 = R + \rho_1 W_1 + \rho_2 W_2 + \rho_1 W_2$$

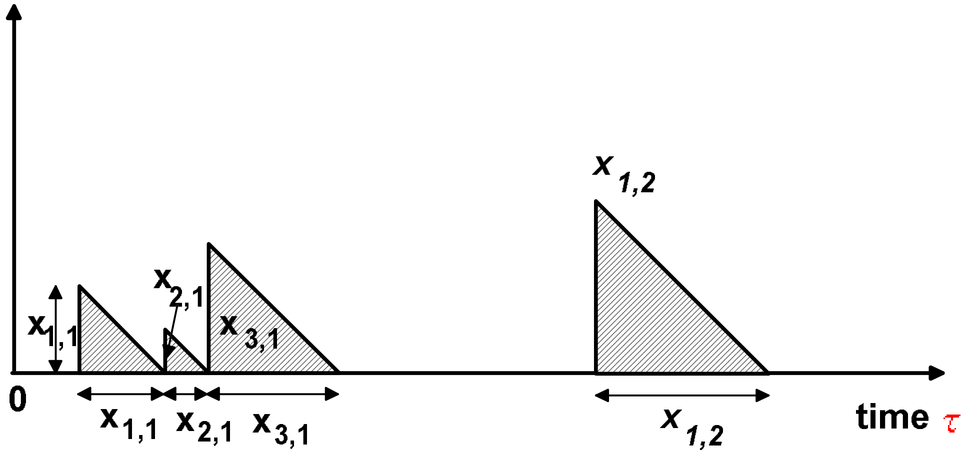
$$\boxed{\Rightarrow W_2 = \frac{R}{(1-\rho_1)(1-\rho_1-\rho_2)}}$$

**For class *k*:**

$$\boxed{W_k = \frac{R}{(1-\rho_1-.....-\rho_{k-1})(1-\rho_1-.....-\rho_k)}}$$

$$T_k = \frac{1}{\mu_k} + W_k$$

$r(\tau)$= residual time for packet in service




$M_k(t) =$ **# customers of class** $k$ **served up to** $t$

**Time average of** $r(\tau)$ **up to time** $t$

$$= \frac{1}{t}\int_0^t r(\tau)d\tau$$

$$= \frac{1}{t}\left[\sum_{i=1}^{M_1(t)} \frac{X_{i,1}^2}{2} + \sum_{i=1}^{M_2(t)} \frac{X_{i,2}^2}{2} + \cdots + \sum_{i=1}^{M_n(t)} \frac{X_{i,n}^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{M_1(t)}{t} \cdot \frac{\sum_{i=1}^{M_1(t)} X_{i,1}^2}{M_1(t)} + \frac{M_2(t)}{t} \cdot \frac{\sum_{i=1}^{M_2(t)} X_{i,2}^2}{M_2(t)} + \cdots + \frac{M_n(t)}{t} \cdot \frac{\sum_{i=1}^{M_n(t)} X_{i,n}^2}{M_2(t)}\right]$$

$$\overset{t\to\infty}{\Rightarrow}$$

$$R = \left[\tfrac{1}{2}\lambda_1 E(X_1^2) + \tfrac{1}{2}\lambda_2 E(X_2^2) + \cdots\cdots + \tfrac{1}{2}\lambda_n E(X_n^2)\right]$$

$$W_k = \frac{\sum_{i=1}^{n} \lambda_i E(X_i^2)}{2(1-\rho_1-\ldots\ldots-\rho_{k-1})(1-\rho_1-\ldots\ldots-\rho_k)}$$

**For stability:** $\rho_1 + \rho_2 + \cdots + \rho_n < 1$ **and** $E(X_i^2) < \infty$

# Preemptive resume priority system

**As before, but customer in service is interrupted by higher-priority customer, and resumes once all higher-priority customers have been served**

**Note: customers of class k are not affected by customers of classes** *k+1*, *k+2*, **...,** *n*

$$\underbrace{T_k}_{} = \frac{1}{\mu_k} + \underbrace{\frac{R_k}{1-\rho_1-.....-\rho_k}}_{} + \underbrace{\sum_{i=1}^{k-1} \frac{1}{\mu_i}\lambda_i T_k}_{}$$

**for priority class** *k* **.**

**waiting time of** *M/G/1* **queue if only customers of classes** *1* **to** *k* **are present, time to serve customer of priorities** *1* **through** *k* **already in the system.**

**time to serve customers of priorities** *1* **through** *k-1* **that arrive while the class** *k***-customer is in queue or in service.**

**where** $R_k = \frac{\sum_{i=1}^{k} \lambda_i \overline{X_i^2}}{2}$

**Finally,** $T_k = \frac{\frac{1}{\mu_k}(1-\rho_1-.....-\rho_k)+R_k}{(1-\rho_1-.....-\rho_{k-1})(1-\rho_1-.....-\rho_k)}$ , $\rho_i = \frac{\lambda_i}{\mu_i}$